NEGATIVE EXPONENTS

The mass of a proton (the positively charged particle in the nucleus of every atom in the universe) is written in *scientific notation* as

$$\mathbf{1.67 \times 10^{-27}} \textbf{ kilograms}$$

How do we interpret that? Is that a really big or a really small number? In this chapter, we will learn exactly what a negative exponent means.

❑ *REVIEW OF THE FIVE LAWS OF EXPONENTS*

Before we begin the discussion of negative exponents, it would be beneficial for us to review the meaning of an exponent and the Five Laws of exponents:

If n is a natural number ($n = 1, 2, 3, \ldots$), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

Now we recall the Five Laws of Exponents. The order doesn't make any difference, but I personally refer to the first one as the First Law of Exponents.

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

❑ DEVELOPING THE MEANING OF A NEGATIVE EXPONENT

Consider the following power of x:

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot \mathbf{x^3} = x^{-3+3} = x^0 = 1$$

which implies that

$$x^{-3} \cdot x^3 = 1$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3:

$$\frac{x^{-3} \cdot x^3}{x^3} = \frac{1}{x^3}$$

$$\Rightarrow \quad \boxed{x^{-3} = \frac{1}{x^3}}$$

We can do the same thing in general. Consider x^{-n}, where n is any number at all:

$$x^{-n} \cdot x^n = x^{-n+n} = x^0 = 1 \qquad \Rightarrow \qquad x^{-n} = \frac{1}{x^n}$$

In short,

$$x^{-n} = \frac{1}{x^n}$$

for ANY number n, assuming $x \neq 0$.

This should convince you that a negative exponent means **reciprocal**. In fact, the exponent doesn't have to be a negative whole number. [Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals; in fact, $w^{-5/4} = \dfrac{1}{w^{5/4}}$ and $h^{-\sqrt{2\pi}} = \dfrac{1}{h^{\sqrt{2\pi}}}$.]

This fact, together with our knowledge of fractions and the Laws of Exponents, is all we need to understand the examples in this chapter.

EXAMPLE 1:

A. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

B. $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$

C. $-3^{-4} = -\dfrac{1}{3^4} = -\dfrac{1}{81}$

See the difference?

D. $(-3)^{-4} = \dfrac{1}{(-3)^4} = \dfrac{1}{81}$

E. $\dfrac{1}{2^{-5}} = \dfrac{1}{\frac{1}{2^5}} = \dfrac{1}{\frac{1}{32}} = \dfrac{1}{1} \cdot \dfrac{32}{1} = 32$

F. $\dfrac{1}{10^{-3}} = \dfrac{1}{\dfrac{1}{10^3}} = \dfrac{1}{1}\cdot\dfrac{10^3}{1} = 10^3 = 1000$

G. $\left(\dfrac{2}{3}\right)^{-3} = \dfrac{1}{\left(\dfrac{2}{3}\right)^3} = \dfrac{1}{\dfrac{8}{27}} = \dfrac{1}{1}\cdot\dfrac{27}{8} = \dfrac{27}{8}$

H. $0^{-5} = \dfrac{1}{0^5} = \dfrac{1}{0} =$ **Undefined**

Homework

1. Evaluate each expression:

a. 3^{-2} b. 2^{-5} c. 10^{-3} d. 5^{-4} e. 1^{-10}

f. -2^{-3} g. $(-3)^{-3}$ h. -10^{-4} i. $(-5)^{-1}$ j. -6^{-2}

k. $\dfrac{1}{3^{-2}}$ l. $\dfrac{1}{10^{-4}}$ m. $\dfrac{1}{5^{-3}}$ n. $\left(\dfrac{3}{4}\right)^{-2}$ o. $\left(\dfrac{1}{6}\right)^{-3}$

2. Explain why $x = 0$ is NOT allowed in the expression x^{-8}.

EXAMPLE 2:

A. $n^{-1} = \dfrac{1}{n}$

B. $a^{-12} = \dfrac{1}{a^{12}}$

C. $\dfrac{x}{y^{-3}} = \dfrac{x}{\frac{1}{y^3}} = x \cdot \dfrac{y^3}{1} = xy^3$

D. $\dfrac{x^{-7}}{a^{12}} = \dfrac{\frac{1}{x^7}}{a^{12}} = \dfrac{1}{x^7} \cdot \dfrac{1}{a^{12}} = \dfrac{1}{a^{12}x^7}$

E. $a^2b^{-3} = a^2 \cdot \dfrac{1}{b^3} = \dfrac{a^2}{b^3}$

F. $R^{-1}T^{-7} = \dfrac{1}{R} \cdot \dfrac{1}{T^7} = \dfrac{1}{RT^7}$

EXAMPLE 3:

A. $x + y^{-5} = x + \dfrac{1}{y^5} = \underbrace{\left[\dfrac{y^5}{y^5}\right]\dfrac{x}{1}}_{\text{to make a common denominator}} + \dfrac{1}{y^5} = \dfrac{xy^5}{y^5} + \dfrac{1}{y^5} = \dfrac{xy^5 + 1}{y^5}$

B. $a^{-2} + b^{-7} = \dfrac{1}{a^2} + \dfrac{1}{b^7} = \dfrac{1}{a^2}\left[\dfrac{b^7}{b^7}\right] + \dfrac{1}{b^7}\left[\dfrac{a^2}{a^2}\right] = \dfrac{b^7 + a^2}{a^2b^7}$

Homework

3. Simplify each expression (no negative exponents in the answer):

a. x^{-9} b. a^{-30} c. $\dfrac{a}{b^{-2}}$ d. $\dfrac{x^{-4}}{y^2}$

e. $\dfrac{w^3}{x^{-3}}$ f. $\dfrac{m^{-3}}{n^{-4}}$ g. $a^{-4}b^{-5}$ h. k^3p^{-5}

i. $h^{-5}z^{7}$ j. $\dfrac{w^{-5}}{w^{-5}}$ k. $x+y^{-1}$ l. $a^{-3}+b^{2}$

m. $p^{-1}+r^{-1}$ n. $w^{-3}-z^{-2}$ o. $a^{-3}-a^{-3}$ p. $a^{-1}b^{-2}c^{-3}$

EXAMPLE 4: **Using the Five Laws of Exponents:**

A. $x^{-6}x^{-8} = x^{-14} = \boldsymbol{\dfrac{1}{x^{14}}}$

Add the exponents

B. $\left(N^{-5}\right)^{-7} = \boldsymbol{N^{35}}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \dfrac{1}{a^4}\cdot\dfrac{1}{b^4}\cdot\dfrac{1}{c^4} = \boldsymbol{\dfrac{1}{a^4b^4c^4}}$

Apply exponent to
each factor

D. $\dfrac{w^5}{w^{15}} = w^{5-15} = w^{-10} = \boldsymbol{\dfrac{1}{w^{10}}}$

Subtract the exponents

E. $\dfrac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \boldsymbol{\dfrac{1}{x^5}}$

Subtract the exponents

F. $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{a^{-3}}{b^{-3}} = \dfrac{\frac{1}{a^3}}{\frac{1}{b^3}} = \dfrac{1}{a^3}\cdot\dfrac{b^3}{1} = \boldsymbol{\dfrac{b^3}{a^3}}$

Apply exponent to
top and bottom

Another way: $\left(\frac{a}{b}\right)^{-3} = \frac{1}{\left(\frac{a}{b}\right)^3} = \frac{1}{\frac{a^3}{b^3}} = \frac{b^3}{a^3}$

G. $(x+y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of two terms — it is not a product — so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x+y)^{-2} = \frac{1}{(x+y)^2} = \frac{1}{(x+y)(x+y)} = \mathbf{\frac{1}{x^2+2xy+y^2}}$$

Homework

4. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^7$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^3$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x+y)^{-1}$ l. $(a-b)^{-2}$

m. $\frac{x^{-5}}{x^3}$ n. $\frac{z^5}{z^{-4}}$ o. $\frac{a^{-5}}{a^{-8}}$ p. $\frac{y^{-12}}{y^{-7}}$

q. $\left(\frac{a}{b}\right)^{-4}$ r. $\left(\frac{y}{x}\right)^{-1}$ s. $\left(\frac{a}{b+c}\right)^{-2}$ t. $\left(\frac{a+b}{xy+wz}\right)^0$

EXAMPLE 5: **Simplify each expression:**

A. $(2x^{-8})(-5x^{6}) = (2)(-5)x^{-8}x^{6} = -10x^{-2} = -10\left(\frac{1}{x^2}\right) = -\frac{\mathbf{10}}{\boldsymbol{x^2}}$

B. $\left(x^2y^{-3}\right)^4 = \left(x^2\right)^4\left(y^{-3}\right)^4 = x^8y^{-12} = x^8\left(\frac{1}{y^{12}}\right) = \frac{\boldsymbol{x^8}}{\boldsymbol{y^{12}}}$

C. $\left(\frac{a^{-3}}{b^4}\right)^{-5} = \frac{\left(a^{-3}\right)^{-5}}{\left(b^4\right)^{-5}} = \frac{a^{15}}{b^{-20}} = \frac{a^{15}}{\frac{1}{b^{20}}} = \frac{a^{15}}{1}\cdot\frac{b^{20}}{1} = \boldsymbol{a^{15}b^{20}}$

Final Note: In the Introduction, we learned about the mass of a proton. We're now in a position to see just what that number is:

$$1.67\times\mathbf{10^{-27}} = 1.67\times\frac{\mathbf{1}}{\mathbf{10^{27}}} = \frac{1.67}{1}\times\frac{1}{10^{27}} = \frac{1.67}{10^{27}}$$

$$= \frac{1.67}{1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$

$$= \boxed{0.00000000000000000000000000167 \text{ kilograms}}$$

Shortcut: Move the decimal point 27 places to the left.

[If available, check out the chapter on *Scientific Notation* for all the details and shortcuts regarding numbers like 1.67×10^{-27}.]

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

5. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^7)$ b. $(-3y^{-3})(2y^{-5})$ c. $(-4a^3)(a^{-7})$

d. $\left(u^3u^{-5}\right)^{-2}$ e. $\left(w^{-5}w^{-1}\right)^7$ f. $\left(a^{-2}b^{-3}\right)^{-5}$

g. $\left(t^3u^{-4}\right)^3$ h. $\left(\dfrac{x^2}{x^5}\right)^{-2}$ i. $\left(\dfrac{y^{-3}}{y^{-4}}\right)^5$

j. $\left(\dfrac{a^{-2}}{b^5}\right)^{-3}$ k. $\left(\dfrac{c^3}{d^{-4}}\right)^{-5}$ l. $\left(\dfrac{a^3b^{-4}}{x^{-12}z^{-2}}\right)^0$

6. Express each scientific notation number as a regular number:

a. 2.3×10^7 b. 7.11×10^{-5} c. 5.09×10^{-10}

Review Problems

7. Evaluate: a. -2^{-4} b. $(-2)^{-4}$

8. Evaluate: a. $\left(\dfrac{4}{5}\right)^{-3}$ b. $\left(\dfrac{1}{9}\right)^{-1}$

9. Simplify: a. $\dfrac{a^{-2}}{b^{-3}}$ b. $\dfrac{x^{-9}}{x^9}$

10. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$

11. Simplify: a. $(x^{-10})^5$ b. $(z^{-5})^{-5}$

12. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^3y^{10})^{-2}$

13. Simplify: a. $\left(\dfrac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\dfrac{a^3}{b^{-4}}\right)^5$

14. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$

15. Explain why $x = 0$ is NOT allowed in the expression x^{-2}.

16. Express as a regular number: 4.9×10^{-13}

Solutions

1. a. $\dfrac{1}{9}$ b. $\dfrac{1}{32}$ c. $\dfrac{1}{1000}$ d. $\dfrac{1}{625}$ e. 1 f. $-\dfrac{1}{8}$

g. $-\dfrac{1}{27}$ h. $-\dfrac{1}{10{,}000}$ i. $-\dfrac{1}{5}$ j. $-\dfrac{1}{36}$ k. 9 l. 10,000

m. 125 n. $\dfrac{16}{9}$ o. 216

2. If x were 0 in the expression x^{-8}, we would have

$x^{-8} = 0^{-8} = \dfrac{1}{0^8} = \dfrac{1}{0}$, which is Undefined.

3. a. $\dfrac{1}{x^9}$ b. $\dfrac{1}{a^{30}}$ c. ab^2 d. $\dfrac{1}{x^4y^2}$ e. x^3w^3 f. $\dfrac{n^4}{m^3}$

g. $\dfrac{1}{a^4b^5}$ h. $\dfrac{k^3}{p^5}$ i. $\dfrac{z^7}{h^5}$ j. 1 k. $\dfrac{xy+1}{y}$ l. $\dfrac{1+a^3b^2}{a^3}$

m. $\frac{r+p}{pr}$ n. $\frac{z^2-w^3}{w^3z^2}$ o. 0 p. $\frac{1}{ab^2c^3}$

4. a. a^4 b. $\frac{1}{x^{15}}$ c. 1 d. $\frac{1}{m^5w^3}$ e. $\frac{1}{x^{10}}$

f. $\frac{1}{a^{15}}$ g. c^{12} h. $\frac{1}{b^2}$ i. $\frac{1}{a^5x^5}$ j. $\frac{1}{x^3y^3z^3}$

k. $\frac{1}{x+y}$ l. $\frac{1}{a^2-2ab+b^2}$ m. $\frac{1}{x^8}$ n. z^9

o. a^3 p. $\frac{1}{y^5}$ q. $\frac{b^4}{a^4}$ r. $\frac{x}{y}$ s. $\frac{b^2+2bc+c^2}{a^2}$

t. 1

5. a. $6x^3$ b. $-\frac{6}{y^8}$ c. $-\frac{4}{a^4}$ d. u^4 e. $\frac{1}{w^{42}}$ f. $a^{10}b^{15}$

g. $\frac{t^9}{u^{12}}$ h. x^6 i. y^5 j. a^6b^{15} k. $\frac{1}{c^{15}d^{20}}$ l. 1

6. a. 23,000,000 b. 0.0000711 c. 0.0000000000509

7. a. $-\frac{1}{16}$ b. $\frac{1}{16}$ **8.** a. $\frac{125}{64}$ b. 9

9. a. $\frac{b^3}{a^2}$ b. $\frac{1}{x^{18}}$ **10.** a. $\frac{1}{a^5}$ b. $\frac{a-1}{a^3}$

11. a. $\frac{1}{x^{50}}$ b. z^{25} **12.** $\frac{1}{48x^3y^{18}}$

13. a. $\frac{1}{x^4}$ b. $a^{15}b^{20}$ **14.** a. $\frac{1}{u^2w^2}$ b. $\frac{1}{u^2+2uw+w^2}$

15. Because $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ which is Undefined.

16. 0.00000000000049

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."

– **George Bernard Shaw**